The Intersection of a Cone and a Sphere: A Contribution to the Geometry of Satellite Viewing

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20 November 1969 and 16 February 1971

There are two geometric problems involved in viewing the earth (or any planet) from a satellite which are common to all instrumental systems: the instantaneous surface seen by an instrument, and the movement of the area seen across the earth's surface. Most commonly, the field of view of an instrument is a cone. Also, a common scanning system involves rotating the viewing axis of the instrument about a given axis, thus tracing out a cone. Frequently then, the geometry of viewing the earth from a satellite requires a solution of the intersection of a sphere and a cone which has an arbitrary orientation relative to the sphere. While this is not in principle a very difficult problem, complications may arise as a result of the initial definitions. Choice of an initial coordinate system centered on the sphere (which is the obvious choice for most practical problems) results in a very long and involved analysis.

This problem has probably been solved many times, at least implicitly, since many of the investigators of satellite observations require knowing scan lines, fields of view, etc. However, to the knowledge of the writer, an explicit solution is not available in the literature. As an aid to workers in the field (especially since certain choices of coordinates may involve considerable mathematical and computational effort), this note is intended to present the equations for the intersection, and to develop the relationships for expressing the intersection curves in geographic coordinates.

Consider a right-handed Cartesian coordinate system, with its origin at the apex of the cone, and the Z axis coincident with the cone axis (see Fig. 1). The half-angle of the cone is θ. The center of the sphere, of radius R, is located in the X,Z plane, at a distance $R+H$ from the origin.

The line joining the sphere center and the origin makes an angle η with the Z axis. The equation of the cone is

$$X^2+Y^2=Z^2\tan^2\theta, \tag{1}$$

while the equation of the sphere is

$$[X+(R+H)\sin\eta]^2+Y^2+[Z-(R+H)\cos\eta]^2=R^2. \tag{2}$$

Substituting Y^2 from (1) into (2) and solving for X in terms of Z yields

$$X=\frac{1}{\sin\eta}\left[-\frac{Z^2}{2(R+H)\cos^2\theta}+Z\cos\eta-\frac{(R+H)^2-R^2}{2(R+H)}\right]. \tag{3}$$

Eq. (3) is the general equation for the intersection. Since it is desired to express the solution in geographic coordinates, Eq. (3) must be referred to a sphere-centered coordinate system oriented to an identifiable geographic location. The only point in the general system which may be so identified is the cone apex (satellite position). This transformation may be accomplished by first translating the origin to the sphere center and reversing the direction of the Z axis (see Fig. 1); thus,

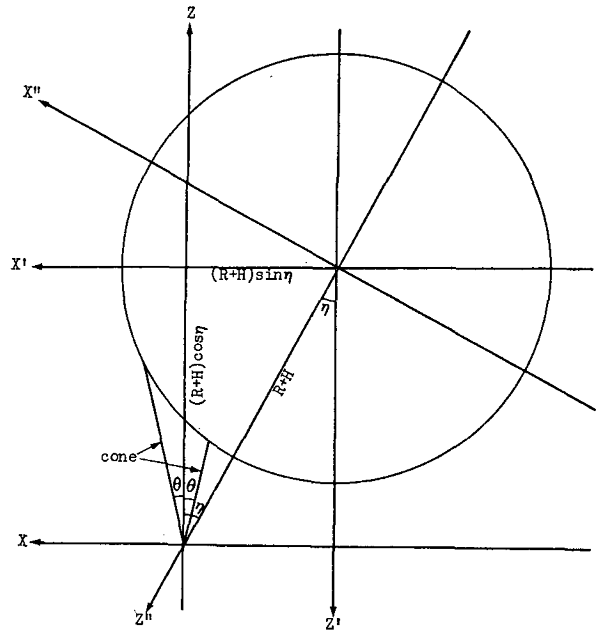

FIG. 1. Coordinate systems.

we have

$$\left.\begin{aligned} X' &= X+(R+H)\sin\eta \\ Y' &= Y \\ Z' &= -Z+(R+H)\cos\eta \end{aligned}\right\}. \tag{4}$$

The system is next rotated about the Y' axis through an angle η, so that the cone apex lies on the Z axis, i.e.,

$$\left.\begin{aligned} X'' &= X'\cos\eta - Z'\sin\eta \\ Y'' &= Y' \\ Z'' &= X'\sin\eta + Z'\cos\eta \end{aligned}\right\}. \tag{5}$$

Combining Eqs. (3), (4) and (5) gives the solution for the intersection in the new coordinate system:

$$X''=(1/\sin\eta)\{[Z''-(R+H)]\cos\eta \pm[(R+H)^2+R^2-2(R+H)Z'']^{\frac{1}{2}}\cos\theta\}, \tag{6}$$

where the plus sign is used for the physically meaningful solution; Y'' may be obtained from the equation of the sphere

$$Y''=[R^2-X''^2-Z''^2]^{\frac{1}{2}}. \tag{7}$$

When $\eta=0$ (i.e., when the cone axis is vertical), X'' is not uniquely given by Eq. (6). The circular intersection curve may be obtained by first determining the constant value of Z'', Z_0''. Multiplying both sides of (6) by $\sin\eta$, letting $\eta=0$, and solving for Z_0'', we have

$$Z_0''=(R+H)\sin^2\theta\pm[R^2-(R+H)^2\sin^2\theta]^{\frac{1}{2}}\cos\theta, \tag{8}$$

where the positive value of the radical must be used. The equation of the intersection circle is then, from the equation of the sphere,

$$X''^2+Y''^2=R^2-Z_0''^2. \tag{9}$$

It is most convenient to perform the actual calculations of the points using Eqs. (6) and (7) [or (8) and (9)], and then to determine the geographic coordinates on a point-by-point basis.

It is obvious that a solution will occur only within a certain range of Z''. The limiting values, Z''_L, are the values of Z'' of the two intersection points which occur in the X'', Z'' plane (see Fig. 2). The two lines which form the intersection of the cone with the X'', Z'' plane will clearly make angles with the Z'' axis of $\eta-\theta$ for the side of the cone nearest the Z'' axis, and $\eta+\theta$ for the far side. In both cases the geometric relationships remain the same. Thus,

$$\sin\beta=(1/R)(R+H)\sin(\eta\pm\theta), \tag{10}$$

$$\gamma=180°-\beta-(\eta\pm\theta), \tag{11}$$

$$Z''_L=R\cos\gamma. \tag{12}$$

Combining these three equations gives

$$Z''_L=(R+H)\sin^2(\eta\pm\theta) +[R^2-(R+H)^2\sin^2(\eta\pm\theta)]^{\frac{1}{2}}\cos(\eta\pm\theta). \tag{13}$$

If part of the cone does not intersect the sphere, a real

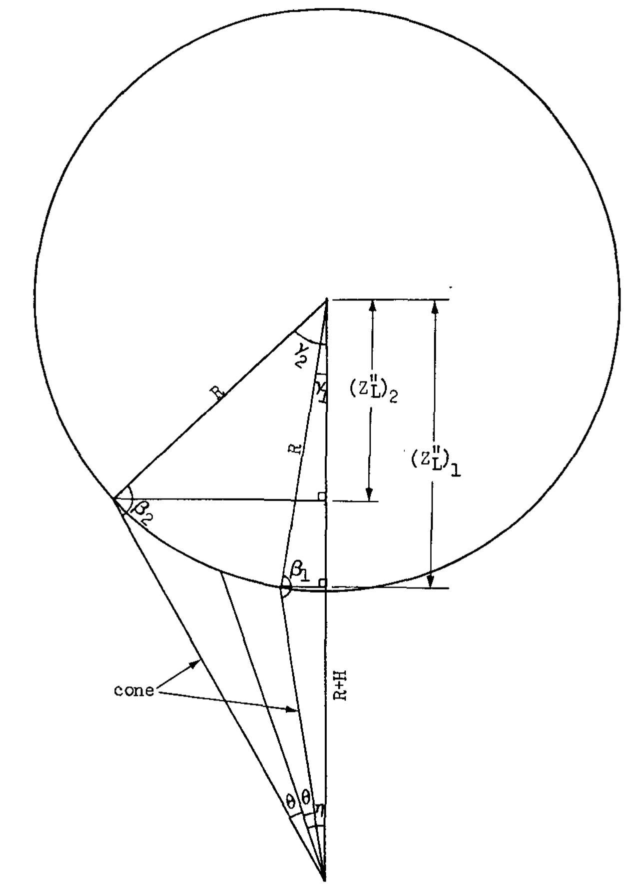

FIG. 2. Limits of intersection.

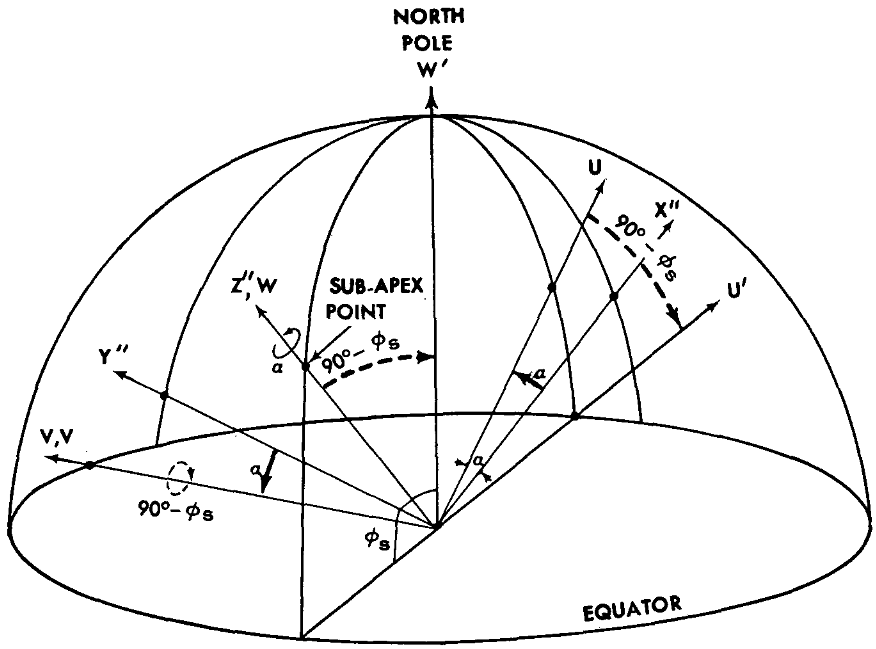

FIG. 3. Rotations for obtaining geographic coordinates.

value will not be obtained for Z''_L for the far edge $(\eta+\theta)$, since the expression under the radical becomes negative. In this case, the limit of the solution is the horizon, as seen from the cone apex, and

$$Z''_L = R^2/(R+H). \tag{14}$$

The values of intersection points relative to geographic axes may be easily obtained by two successive coordinate rotations. The azimuth from north to the cone axis α and the latitude ϕ_S of the sub-apex point, must be known. The first rotation (see Fig. 3) is through an angle α about the Z'' axis, giving

$$\left.\begin{aligned} U &= Y'' \sin\alpha + X'' \cos\alpha \\ V &= Y'' \cos\alpha - X'' \sin\alpha \\ W &= Z'' \end{aligned}\right\}. \tag{15}$$

This rotation causes the U,W plane to be coincident with the meridional plane containing the cone apex.

The second rotation (see Fig. 3) is about the V axis, through an angle $90°-\phi_S$. In this case we have

$$\left.\begin{aligned} U' &= U \sin\phi_S - W \cos\phi_S \\ V' &= V \\ W' &= U \cos\phi_S + W \sin\phi_S \end{aligned}\right\}. \tag{16}$$

The W' axis is now coincident with the earth's axis, and the U',V' plane is coincident with the earth's equatorial plane. The cone apex remains in the U',W' plane, located at a negative value of U'.

The latitude ϕ, of the intersection point, is given by

$$\sin\phi = W'/R, \tag{17}$$

and the longitude λ, relative to the longitude of the apex, by

$$\sin\lambda = V'/[U'^2+V'^2]^{\frac{1}{2}}. \tag{18}$$

The quadrant of λ is determined by means of the signs of U' and V'. The true longitude may then be obtained by adding to λ the longitude of the cone apex.

Finally, it might be mentioned that in addition to determining the field of view of a satellite instrument, or of calculating scan paths, there are additional applications. For example, the coordinates of a small circle centered on any point on the earth may be obtained by using the vertical case $(\eta=0)$, while the coordinates of a great circle passing through any point, and having any azimuth at that point, are determined by letting $\eta=90°$ and $\theta=90°$ (i.e., a vertical plane).